The inner spine strategy is normalising for distributive λ-calculus

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Abstract

We positively answer Question A.1.6 of Klop's Ustica notes [3, p.81]:

`Is there a recursive normalizing one-step reduction strategy for micro λ-calculus?`

where micro λ-calculus refers to an implementation of the λ-calculus due to Révész [4], implementing β-reduction by means of 'micro steps' recursively distributing a β-redex $(\lambda x.M)N$ over its body M.

1 Answer

Definition 1. Distributive *reduction*[1] $\rightarrow$ *on λ-terms is generated by:*

$$
\begin{array}{rcl}
(\lambda x.x)N & \rightarrow & N \\
(\lambda x.y)N & \rightarrow & y \quad \textit{for } x \neq y \\
(\lambda xy.M)N & \rightarrow & \lambda y.(\lambda x.M)N \quad \textit{for } x \neq y \textit{ and } y \notin N \\
(\lambda x.M_1 M_2)N & \rightarrow & (\lambda x.M_1)N((\lambda x.M_2)N)
\end{array}
$$

Remark that a term is a distributive redex if and only if it is β-redex, hence distributive and β-normal forms coincide.

Normalisation of our strategy answering the question in the abstract, relies on the one hand on normalisation of spine reductions for the ordinary λ-calculus, and on the other hand on termination of pure distribution steps, as encountered in the λ-calculus with explicit substitutions λx.

Definition 2. *An* inner *spine strategy always contracts an innermost redex among the spine redexes [1, Definition 4.7(i)].*

By the above remark, the spine redexes w.r.t. distributive reduction coincide with those for ordinary β-reduction.

If M distributively rewrites to M', then in general M need not β-rewrite to M', but M and M' are β-convertible:

[1] Distributive reduction is our attempt to provide 'micro λ-calculus' with a more systematic name.

- $(\lambda x.x)N \to_\beta x[x{:=}N] = N$;
- $(\lambda x.y)N \to_\beta y[x{:=}N] = y$;
- $(\lambda xy.M)N \to_\beta (\lambda y.M)[x{:=}N] = \lambda y.M[x{:=}N] \leftarrow_\beta \lambda y.(\lambda x.M)N$;
- $(\lambda x.M_1 M_2)N \to_\beta (M_1 M_2)[x{:=}N] = M_1[x{:=}N]M_2[x{:=}N] \leftarrow_\beta$ $(\lambda x.M_1)N((\lambda x.M_2)N)$.

Our strategy relies on the observation that distributive reduction *is* preserved when projecting every term M to its full-β-development $M^\bullet$, as long as the steps of the former are not β-destructive. Here the *full-β-development* $M^\bullet$ of a term M is the term obtained by β-contracting all redexes of M, and a step is called *destructive*, if the redex contracted is of shape $(\lambda x.(\lambda y.M_1)M_2)N$, i.e. in case of distribution of N over an application $(\lambda y.M_1)M_2)$ which itself is a redex. Non-destructive steps will be mapped to *β-reduction sequences* by $\bullet$.

Instead of proving this general fact, we note inner spine steps are non-destructive by innerness, and show that each such inner spine step is mapped to at most a *single β-reduction step* by $\bullet$. Moreover, in case a distributive inner spine step is mapped to the empty step by $\bullet$, i.e. if it is erased, then that step did not *create* a redex, hence it was a purely distributive step. This can be expressed formally by mapping the step to an x-step in Bloo and Rose's λ-calculus with explicit substitutions λx [2], via an explicification map $\diamond$. Here, the *explicification* $M^\diamond$ of a term M is obtained by replacing each redex $(\lambda y.P)Q$ by the redex $P\langle y{:=}Q\rangle$ in the λ-calculus with explicit substitutions λx.

Lemma 3. *If $M \to N$ is an inner spine step, then $M^\bullet \to_\beta N^\bullet$ by a spine step, or $M^\bullet = N^\bullet$ and $M^\diamond \to_x N^\diamond$.*

Proof. See Appendix A. □

Theorem 4. *Inner spine strategies are normalising.*

Proof. By the lemma, an infinite distributive reduction from some term M having a normal form $\hat{M}$, would give rise to an infinite spine β-reduction from $M^\bullet$, unless from some moment N on in the distributive reduction all further terms are mapped to $N^\bullet$. But then by the lemma again, the infinite distributive reduction from N would give rise to an infinite x-reduction from $N^\diamond$.

Infinite spine β-reductions are impossible from $M^\bullet$ since M and $M^\bullet$ are β-convertible, hence have the same β-normal form $\hat{M}$, and spine strategies are needed strategies, hence normalising [1].

Infinite $\to_x$-reductions are impossible since $\to_x$ reduction (the substitution rules) is known to be terminating for the λx-calculus [2]. □

The strategy can be made effective by first searching for the leftmost path containing a redex, and then taking the innermost redex on that path.

The essence of our strategy is to avoid destruction of redexes. In particular, the inner spine strategy avoids (by innerness) that distribution of the outer redex in $Q = (\lambda x.(\lambda y.M)N)P$ destroys the inner one, thereby blocking Klop's counterexample to preservation of strong normalisation for distributive reduction. In a way, this demonstrates that Klop's spiralling reduction from Q is the only extra 'cause' for non-termination in the implementation of β-reduction by means of distributive reduction. We expect it to be be easy to adapt our proof to other strategies breaking the spiral.

References

[1] H.P. Barendregt, J.R. Kennaway, J.W. Klop, and M.R. Sleep. Needed reduction and spine strategies for the lambda calculus. *Information and Computation*, 75(3):191–231, December 1987.

[2] C.J. Bloo. *Preservation of Termination for Explicit Substitution.* PhD thesis, Technische Universiteit Eindhoven, October 2, 1997.

[3] J.W. Klop. Term rewriting systems, September 1985. Notes prepared for the seminar on Reduction Machines, organized by C. Böhm, Ustica.

[4] G. Révész. Axioms for the theory of lambda-conversion. *SIAM Journal on Computing*, 14(2):373–382, May 1985.

A Proof of Lemma 3

We provide a detailed proof of Lemma 3, stating that if $M \to N$ is an inner spine step, then $M^\bullet \to^\beta N^\bullet$ by a spine step, or $M^\bullet = N^\bullet$ and $M^\diamond \to_x N^\diamond$.

Proof. By induction on the generation of steps.

- If the step is due to an instance $\ell \to r$ of one of the distributive rule schemata, then $\ell^\bullet = r^\bullet$:
  - $((\lambda x.x)N)^\bullet = x[x{:=}N^\bullet] = N^\bullet$;
  - $((\lambda x.y)N)^\bullet = y[x{:=}N^\bullet] = y = y^\bullet$;
  - $((\lambda xy.M)N)^\bullet = (\lambda y.M^\bullet)[x{:=}N^\bullet] =$ $\lambda y.M^\bullet[x{:=}N^\bullet] = \lambda y.((\lambda x.M)N)^\bullet = (\lambda y.(\lambda x.M)N))^\bullet$;
  - $((\lambda x.M_1M_2)N)^\bullet = (M_1M_2)^\bullet[x{:=}N^\bullet] = (M_1^\bullet M_2^\bullet)[x{:=}N^\bullet] =$ $M_1^\bullet[x{:=}N^\bullet]M_2^\bullet[x{:=}N^\bullet] = ((\lambda x.M_1)N)^\bullet((\lambda x.M_2)N)^\bullet =$ $((\lambda x.M_1)N((\lambda x.M_2)N))^\bullet$, which holds by the step being inner spine, guaranteeing that the application M_1M_2 is not a redex, hence that $(M_1M_2)^\bullet = M_1^\bullet M_2^\bullet$.

  We show $\ell^\diamond \to_x r^\diamond$ holds in each case:

  - $((\lambda x.x)N)^\diamond = x\langle x{:=}N^\diamond\rangle \to_x N^\diamond$;
  - $((\lambda x.y)N)^\diamond = y\langle x{:=}N^\diamond\rangle \to_x y = y^\diamond$;

– $((\lambda xy.M)N)^\diamond = (\lambda y.M^\diamond)\langle x{:=}N^\diamond\rangle \to_x \lambda y.M^\diamond\langle x{:=}N^\diamond\rangle = \lambda y.((\lambda x.M)N)^\diamond = (\lambda y.(\lambda x.M)N))^\diamond$;
– $((\lambda x.M_1M_2)N)^\diamond = (M_1M_2)^\diamond\langle x{:=}N^\diamond\rangle = (M_1^\diamond M_2^\diamond)\langle x{:=}N^\diamond\rangle \to_x M_1^\diamond\langle x{:=}N^\diamond\rangle M_2^\diamond\langle x{:=}N^\diamond\rangle = ((\lambda x.M_1)N)^\diamond((\lambda x.M_2)N)^\diamond = ((\lambda x.M_1)N((\lambda x.M_2)N))^\diamond$, which again holds as before by the step being inner spine, guaranteeing $(M_1M_2)^\diamond = M_1^\diamond M_2^\diamond$.

- If the inner spine step is $MP \to NP$ due to $M \to N$, then by the induction hypothesis $M^\bullet \to_\beta N^\bullet$ by a spine step, or $M^\bullet = N^\bullet$ and $M^\diamond \to_x N^\diamond$, and we distinguish cases on whether MP is a redex or not.

  If MP is a redex, then for some M', $M = \lambda x.M'$ so $M^\bullet = \lambda x.M'^\bullet$, hence by the shape of the rules for some N', $N = \lambda x.N'$ so $N^\bullet = \lambda x.N'^\bullet$. Hence if $M^\bullet \to_\beta N^\bullet$ by a spine step, then $M'^\bullet \to_\beta N'^\bullet$ by a spine step. Since MP is itself a redex, a spine redex in M must occur on the left spine of M, and by closure of both this property and $\to_\beta$ under substitution $(MP)^\bullet = M'^\bullet[x{:=}P^\bullet] \to_\beta^= N'^\bullet[x{:=}P^\bullet] = (NP)^\bullet$. If $M^\bullet = N^\bullet$ and $M^\diamond \to_x N^\diamond$, then $M'^\bullet = N'^\bullet$ so $(MP)^\bullet = M'^\bullet[x{:=}P^\bullet] = N'^\bullet[x{:=}P^\bullet] = (NP)^\bullet$, and $M'^\diamond \to_x N'^\diamond$ so $(MP)^\diamond = M'^\diamond\langle x{:=}P^\diamond\rangle \to_x N'^\diamond\langle x{:=}P^\diamond\rangle = (NP)^\diamond$.

  If MP is not a redex, then $MP^\bullet = M^\bullet P^\bullet$ and $MP^\diamond = M^\diamond P^\diamond$, and we distinguish further cases on whether NP is a redex or not.

  If NP is a redex, then for some N', $N = \lambda y.N'$, hence by the shape of the rules the step $M \to N$ must be an instance of either the first or the third rule. In case it is an instance of the first rule, then $M = (\lambda x.x)N$ and $M^\bullet P^\bullet = ((\lambda x.x)N)^\bullet P^\bullet = x[x{:=}N^\bullet]^\bullet P^\bullet = N^\bullet P^\bullet = (\lambda y.N')^\bullet P^\bullet = (\lambda y.N'^\bullet)P^\bullet \to_\beta N'^\bullet[y{:=}P^\bullet] = ((\lambda y.N')P)^\bullet = (NP)^\bullet$. In case it is an instance of the third rule, then for some M',N'', $M = (\lambda xy.M')N''$, $N' = (\lambda x.M')N''$, and $(MP)^\bullet = M^\bullet P^\bullet = ((\lambda xy.M')N'')^\bullet P^\bullet = (\lambda y.M')^\bullet[x{:=}N''^\bullet]P^\bullet = (\lambda y.M'^\bullet)[x{:=}N''^\bullet]P^\bullet = (\lambda y.M'^\bullet[x{:=}N''^\bullet])P^\bullet \to_\beta M'^\bullet[x{:=}N''^\bullet][y{:=}P^\bullet] = ((\lambda x.M')N'')^\bullet[y{:=}P^\bullet] = (\lambda y.(\lambda x.M')N'')P^\bullet = ((\lambda y.N')P)^\bullet = (NP)^\bullet$. In both cases we conclude, since head-redexes are spine redexes.

  In case NP is not a redex, $(NP)^\bullet = N^\bullet P^\bullet$ and $(NP)^\diamond = N^\diamond P^\diamond$. If $M^\bullet \to_\beta N^\bullet$ is a spine step, then $(MP)^\bullet = M^\bullet P^\bullet \to_\beta N^\bullet P^\bullet = (NP)^\bullet$ is a spine step as well. If $M^\bullet = N^\bullet$ and $M^\diamond \to_x N^\diamond$, then $(MP)^\bullet = M^\bullet P^\bullet = N^\bullet P^\bullet = (NP)^\bullet$ and $(MP)^\diamond = M^\diamond P^\diamond \to_x N^\diamond P^\diamond = (NP)^\diamond$.

- If the inner spine step is $PM \to PN$ due to $M \to N$, then by the induction hypothesis $M^\bullet \to_\beta N^\bullet$ by a spine step, or $M^\bullet = N^\bullet$ and $M^\diamond \to_x N^\diamond$. By the assumption that steps are inner spine, PM itself is not a redex, hence neither is PN. Hence in the former case $(PM)^\bullet = P^\bullet M^\bullet \to_\beta P^\bullet N^\bullet = (PN)^\bullet$ is a spine step, and in the latter $(PM)^\bullet = P^\bullet M^\bullet = P^\bullet N^\bullet = (PN)^\bullet$ and $(PM)^\diamond = P^\diamond M^\diamond \to_x P^\diamond N^\diamond = (PN)^\diamond$.

- If the step is $\lambda x.M \to \lambda x.N$ due to $M \to N$, $M^\bullet \to_\beta N^\bullet$ by a spine step, or $M^\bullet = N^\bullet$ and $M^\diamond \to_x N^\diamond$. In the former case $(\lambda x.M)^\bullet = \lambda x.M^\bullet \to_\beta \lambda x.N^\bullet = (\lambda x.M)^\bullet$ is a spine step, and in the latter $(\lambda x.M)^\bullet = \lambda x.M^\bullet = \lambda x.N^\bullet = (\lambda x.M)^\bullet$ and $(\lambda x.M)^\diamond = \lambda x.M^\diamond \to_x \lambda x.N^\diamond = (\lambda x.M)^\diamond$. □